Memory-enhanced energetic stability for a fractional oscillator with fluctuating frequency

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The long-time limit behavior of the variance and the correlation function for the output signal of a fractional oscillator with fluctuating eigenfrequency subjected to a periodic force is considered. The influence of a fluctuating environment is modeled by a multiplicative white noise and by an additive noise with a zero mean. The viscoelastic-type friction kernel with memory is assumed as a power-law function of time. The exact expressions of stochastic resonance (SR) characteristics such as variance and signal-to-noise ratio (SNR) have been calculated. It is shown that at intermediate values of the memory exponent the energetic stability of the oscillator is significantly enhanced in comparison with the cases of strong and low memory. A multiresonancelike behavior of the variance and SNR as functions of the memory exponent is observed and a connection between this effect and the memory-induced enhancement of energetic stability is established. The effect of memory-induced energetic stability encountered in case the harmonic potential is absent, is also discussed.

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I. INTRODUCTION

The influence of random perturbations on a dynamical system is a problem of interest in various fields of science and engineering [1,2]. Of particular interest are studies of random parametric vibrations, where some phenomenological stochastic terms describing random environmental loadings are added in the dynamical equations. This idea has led to many important discoveries such as stochastic resonance [3], stochastic ratchets [1,4], and noise-induced multistability as well as phase transitions in some complex systems [5], to name a few. Since the harmonic oscillator is the simplest toy model for different phenomena in nature, linear models of oscillators with multiplicative noise are of particular interest. It is shown that the influence of noise on oscillator eigenfrequency may lead to different resonant phenomena. First, it may cause energetic instability [6–8]. This phenomenon is a stochastic counterpart of classical parametric resonance [7,9]. Second, if the oscillator is subjected to an external periodic force and the fluctuations of the oscillator frequency are colored, the behavior of the moments and signal-to-noise ratio (SNR) of the output signal shows a nonmonotonic dependence on noise parameters, i.e., stochastic resonance (SR) in a wide sense [8,10]. Another popular generalization of the harmonic oscillator, beside inclusion of a multiplicative noise, consists in the replacement of the usual friction term in the dynamical equation for a harmonic oscillator by a generalized friction term with a power-law type memory [11–14]. It has been shown that such fractional-order models provide an excellent instrument for the description of the memory and hereditary properties of various viscoelastic materials and processes [15]. Examples of such systems are supercooled liquids, glasses, colloidal suspensions, dense polymer solutions [16,17], and semiconductors [18]. The fractional-order models have also been successfully used in describing anomalous diffusion phenomena for reaction kinetics and fluoresce intermittency of single enzymes [19], for nuclear fusion reactions [20], and for intrinsic conformational dynamics of proteins [13,21].

Although the behavior of both above-mentioned generalizations of the harmonic oscillator, i.e., a harmonic oscillator with fluctuating frequency and a fractional oscillator, have been investigated in detail (see, e.g., [8,11]), it seems that analysis of the potential consequences of interplay between eigenfrequency fluctuations and memory effects is still missing in literature. This is quite surprising in view of the fact that the importance of fluctuations and viscoelasticity for biological systems, e.g., living cells, has been well recognized [15,22].

Thus motivated, the authors of [23] have recently considered a fractional oscillator with a power-law memory kernel subjected to an external periodic force. The influence of the fluctuating environment was modeled by a multiplicative dichotomous noise (fluctuating eigenfrequency). This model demonstrates that an interplay of colored noise and memory can generate a variety of cooperative effects, such as multiresonance versus the driving frequency and the friction coefficient as well as SR versus noise parameters. However, in this work the authors have confined themselves to investigating the first moment of the output signal and thus the possible influence of energetic instability on the output signal is not considered.

It is of interest, both from theoretical and possible experimental viewpoints, to know the behavior of the second moments of the output signal (such as variance and SNR) in the case of similar model systems.

Inspired by the results of [23], we consider in the present paper a model similar to the one presented in [23], except that the influence of the fluctuating environment is modeled not by a dichotomous noise, but by a multiplicative white noise (fluctuating eigenfrequency) and by an additive fractional noise with a zero mean.

The main contribution of this paper is as follows. In the long-time limit, $t \to \infty$, we provide exact formulas for the analytic treatment of the dependence of SR characteristics, such as SNR and variance of the output signal, on system parameters. Furthermore, we establish the sufficient conditions for the occurrence of energetic instability and analyze

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the behavior of critical noise intensity (i.e., the intensity of a multiplicative noise above which the system is unstable) on the memory exponent. It is found that the energetic stability of the oscillator is significantly enhanced at intermediate values of the memory exponent. Furthermore, we show that in certain parameter regions multiresonance is manifested in the dependence of the output SNR of the noisy fractional oscillator upon the memory exponent as well as on the friction coefficient. Moreover, we demonstrate the phenomenon of memory-induced energetic stability for an unbounded system (the trapping harmonic potential is absent). The last mentioned effect is highly unexpected, because in the case without memory such a system is always energetically unstable.

The structure of the paper is as follows. In Sec. II, we present the model investigated. Exact formulas are found for the analysis of the long-time behavior of the SR characteristics. In Sec. III, we analyze the dependence of the output characteristics on the memory exponent and on the friction coefficient. Sec. IV contains some brief concluding remarks. Some formulas are delegated to the Appendix.

II. MODEL AND THE EXACT MOMENTS

A. Model

As a model for an oscillatory system with memory, strongly coupled with a noisy environment, we consider a fractional oscillator with a fluctuating eigenfrequency

$$\ddot{X}(t) + \gamma \frac{d^\alpha X}{dt^\alpha} + [\omega^2 + Z(t)]X(t) = \xi(t) + A_0 \sin(\Omega t), \quad (1)$$

where $\dot{X} \equiv dX/dt$, $X(t)$ is the oscillator displacement, γ is a friction constant, and A_0 and Ω are the amplitude and the frequency of the external harmonic driving force, respectively, and the operator d^α/dt^α with $0<\alpha<1$ denotes the fractional derivative in Caputo's sense, given by [24]

$$\frac{d^\alpha X}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{X}(t')}{(t-t')^\alpha} dt', \quad (2)$$

$\Gamma(y)$ being the gamma function. Fluctuations of the eigenfrequency ω are expressed as a Gaussian white noise $Z(t)$ with a zero mean and a delta-correlated correlation function:

$$\langle Z(t) \rangle = 0, \quad \langle Z(t) Z(t') \rangle = 2D\delta(t-t'), \quad (3)$$

where D is the noise intensity. The zero-centered random force $\xi(t)$ with a stationary correlation function

$$C(|t-t'|) \coloneqq \langle \xi(t)\xi(t') \rangle, \quad \langle \xi(t) \rangle = 0 \quad (4)$$

is assumed as statistically independent from the noise $Z(t)$. Depending on the physical situation, the driving noise $\xi(t)$ can be regarded either as an internal noise, in which case its stationary correlation function satisfies Kubo's second fluctuation-dissipation theorem [25] expressed as

$$C(|\tau|) = \frac{k_B T \gamma}{\Gamma(1-\alpha)|\tau|^\alpha} \quad (5)$$

(here T is the absolute temperature of the heat bath, and k_B is the Boltzmann constant), or as an external noise, in which case the driving noise $\xi(t)$ and the dissipation may have different origins and no fluctuation-dissipation relation holds.

B. First moments

The second-order differential Eq. (1) can be written as two first-order differential equations

$$\dot{X}(t) = Y(t), \quad (6)$$

$$\dot{Y}(t) + \frac{\gamma}{\Gamma(1-\alpha)} \int_0^t \frac{Y(t')}{(t-t')^\alpha} dt' + [\omega^2 + Z(t)]X(t) = \xi(t) + A_0 \sin(\Omega t), \quad (7)$$

which, after averaging over the ensemble of realizations of the random processes $Z(t)$ and $\xi(t)$, take the following form:

$$\langle X(t) \rangle^{\cdot} = \langle Y(t) \rangle,$$

$$\langle Y(t) \rangle^{\cdot} + \frac{\gamma}{\Gamma(1-\alpha)} \int_0^t \frac{\langle Y(t') \rangle}{(t-t')^\alpha} dt' + \omega^2 \langle X(t) \rangle = A_0 \sin(\Omega t). \quad (8)$$

Here we have used that from Eq. (3) it follows that the correlator

$$\langle Z(t) X(t) \rangle = 0. \quad (9)$$

Thus, it turns out that fluctuations of the frequency do not affect the first moments $\langle X(t) \rangle$ and $\langle \dot{X}(t) \rangle$ of the oscillator, provided the fluctuations are delta correlated, and $\langle X(t) \rangle$ remains equal to the noise-free solution. By means of the Laplace transformation to Eqs. (6) and (7) one can easily obtain formal expressions for the oscillator displacement $X(t)$ and the velocity $\dot{X}(t) = Y(t)$ in the following forms:

$$X(t) = \langle X(t) \rangle + \int_0^t H(t-\tau)[\xi(\tau) - X(\tau)Z(\tau)]d\tau, \quad (10)$$

$$Y(t) = \langle Y(t) \rangle + \int_0^t \dot{H}(t-\tau)[\xi(\tau) - X(\tau)Z(\tau)]d\tau, \quad (11)$$

where the averages $\langle X(t) \rangle$ and $\langle Y(t) \rangle$ are given by

$$\langle X(t) \rangle = \dot{x}_0 H(t) + x_0 \left[1 - \omega^2 \int_0^t H(\tau) d\tau \right] + A_0 \int_0^t H(t-\tau) \sin(\Omega \tau) d\tau, \quad (12)$$

$$\langle Y(t) \rangle = \dot{x}_0 \dot{H}(t) - \omega^2 x_0 H(t) + A_0 \int_0^t \dot{H}(t-\tau) \sin(\Omega \tau) d\tau, \quad (13)$$

with deterministic initial conditions such as $X(0) = x_0$ and $Y(0) = \dot{x}_0$. The kernel $H(t)$ with the initial condition $H(0) = 0$ is the Laplace inversion of

$$\hat{H}(s) = \frac{1}{s^2 + \gamma s^\alpha + \omega^2}, \quad (14)$$

where

$$\hat{H}(s) = \int_0^\infty e^{-st} H(t) dt. \quad (15)$$

An integral representation of the relaxation function $H(t)$ is given by Eqs. (A1)–(A6) in the Appendix. Thus, in the long-time limit, $t \to \infty$, the memory about the initial conditions will vanish (cf. [11]) as

$$x_0 \left[1 - \omega^2 \int_0^t H(\tau) d\tau \right] \approx \frac{\gamma x_0}{\omega^2 \Gamma(1-\alpha) t^\alpha} \quad (16)$$

and the average oscillator displacement, $\langle X(t) \rangle_{as} := \langle X(t) \rangle_{|t \to \infty}$, is given by

$$\langle X(t) \rangle_{as} = A_0 \int_0^t H(t-\tau) \sin(\Omega \tau) d\tau. \quad (17)$$

Equation (17) can be written as

$$\langle X(t) \rangle_{as} = A \sin(\Omega t + \varphi). \quad (18)$$

The response A and the phase shift φ are obtained by means of the complex susceptibility

$$\chi = \chi' + i\chi'' = \hat{H}(-i\Omega), \quad (19)$$

where χ' and χ'' are the real and imaginary parts of the susceptibility, respectively.

For the response, we have

$$A^2 = A_0^2 |\chi|^2 = \frac{A_0^2}{[\omega^2 - \Omega^2 + \Omega^\alpha \gamma \cos(0.5\alpha\pi)]^2 + \Omega^{2\alpha} \gamma^2 \sin^2(0.5\alpha\pi)}, \quad (20)$$

and the phase shift can be represented as

$$\varphi = \arctan\left(-\frac{\chi''}{\chi'} \right) = \arctan\left[-\frac{\Omega^\alpha \gamma \sin(0.5\alpha\pi)}{\omega^2 - \Omega^2 + \Omega^\alpha \gamma \cos(0.5\pi\alpha)} \right]. \quad (21)$$

Here we emphasize that the formulas (18), (20), and (21) for a deterministic fractional oscillator have been previously represented in [11].

To avoid misunderstandings, let us mention that in contrast to the model considered in [23], where the dependence of the response A on the noise parameters was significant, in the present model the response A is independent of the noise parameters and remains equal to the noise-free solution.

C. Second moments

Here, we will evaluate the long-time behavior of the variance $\sigma^2(X)$ and the correlation function $K(\tau,t)$ of the oscillator displacement:

$$\sigma^2(t) \equiv \langle [X(t) - \langle X(t) \rangle]^2 \rangle, \quad (22)$$

$$K(\tau,t) \equiv \langle [X(t+\tau) - \langle X(t+\tau) \rangle][X(t) - \langle X(t) \rangle] \rangle. \quad (23)$$

Starting from Eq. (10), we obtain, in the long-time limit, $t \to \infty$, the following asymptotic formula for the correlation function

$$K_{as}(\tau,t) = \int_0^\infty \int_0^\infty H(t_1) H(t_2) C(|\tau + t_2 - t_1|) dt_1 dt_2 + 2D \int_0^\infty H(\tau + t_1) H(t_1) \{\sigma^2(t - t_1) + A^2 \sin^2[\Omega(t - t_1) + \varphi]\} dt_1. \quad (24)$$

Here, we have used the fact that because of the statistical independence of the processes $\xi(t)$ and $Z(t)$, it follows from Eq. (9) that

$$\langle \xi(t_1) X(t_2) Z(t_2) \rangle = \langle \xi(t_2) X(t_1) Z(t_1) \rangle = 0, \quad (25)$$

and that in the case of a delta-correlated noise $Z(t)$ the correlator $\langle X(t_1) X(t_2) Z(t_1) Z(t_2) \rangle$ can be given by

$$\langle X(t_1) X(t_2) Z(t_1) Z(t_2) \rangle = 2D \langle X^2(t_2) \rangle \delta(|t_2 - t_1|). \quad (26)$$

In the limit $t \to \infty$ the two-time correlation function $K_{as}(\tau,t)$ depends on both of the times t and τ and becomes a periodic function of t with the period of the external driving, $\mathcal{T} = 2\pi/\Omega$. Thus, as in [26], we define the one-time correlation function $K_1(\tau)$ as an average of the two-time correlation function over a period of the external driving, i.e.,

$$K_1(\tau) = \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} K_{as}(\tau,t) dt. \quad (27)$$

Using Eq. (24) we obtain

$$K_1(\tau) = D(2\bar{\sigma}^2 + A^2) \int_0^\infty H(\tau + t) H(t) dt + \int_0^\infty \int_0^\infty H(t_1) H(t_2) C(|\tau + t_2 - t_1|) dt_1 dt_2, \quad (28)$$

where $\bar{\sigma}^2$ is the time-homogeneous part of the variance of the oscillator displacement X in the asymptotic regime, $t \to \infty$, i.e.,

$$\bar{\sigma}^2 = \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} \sigma_{as}^2[X(t)] dt. \quad (29)$$

As $K_1(0) = \bar{\sigma}^2$ we find from Eq. (28) that

$$\bar{\sigma}^2 = \frac{1}{D_{cr} - D} \left(\frac{A^2}{2} D + D_{cr} \sigma_0^2 \right), \quad (30)$$

where the critical noise intensity D_{cr} reads

$$D_{cr} = \left[2 \int_0^\infty H^2(t) dt \right]^{-1}, \quad (31)$$

and

$$\sigma_0^2 = \int_0^\infty \int_0^\infty H(t_1)H(t_2)C(|t_2 - t_1|)dt_1 dt_2 \quad (32)$$

is the asymptotic value, $t \to \infty$, of the variance of the oscillator displacement in the case where the fluctuations of the eigenfrequency and the external harmonic force are both absent. From Eq. (30) we can see that the stationary regime is possible only if $D < D_{cr}$. The exact formula useful for a numerical treatment of the critical noise intensity D_{cr} is given by Eq. (A7) (see Appendix). As the noise intensity D tends to the critical value D_{cr} the variance $\bar{\sigma}^2$ increases to infinity. This is an indication that for $D > D_{cr}$ energetic instability appears, which manifests itself in an unlimited increase of second-order moments of the output of the oscillator with time, while the mean value of the oscillator displacement remains finite [6,7]. This phenomenon is a stochastic counterpart of the classical parametric resonance in the case of an ordinary deterministic oscillator (without a memory kernel) with time-dependent frequencies [7,9].

Turning now to Eq. (28), we consider the SNR of the output signal. The one-time correlator

$$K(\tau) = \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} \langle X(t+\tau)X(t)\rangle_{as} dt \quad (33)$$

can be written exactly as the sum of two contributions,

$$K(\tau) = K_1(\tau) + K_2(\tau), \quad K_2(\tau) = \frac{A^2}{2}\cos(\Omega\tau), \quad (34)$$

i.e., the coherent part $K_2(\tau)$, which is periodic in τ with the period $\mathcal{T}$, and the incoherent part $K_1(\tau)$, which decays to zero for large values of τ. According to [26], the output SNR (R_{out}) at the forcing frequency Ω is defined in terms of the Fourier cosine transform of the coherent and incoherent parts of $K(\tau)$. Namely,

$$R_{out} = \frac{\Gamma_2}{\Gamma_1}, \quad (35)$$

where

$$\Gamma_2 = \frac{2}{\mathcal{T}}\int_0^{\mathcal{T}} K_2(\tau)\cos(\Omega\tau)d\tau = \frac{A^2}{2}, \quad (36)$$

and

$$\Gamma_1 = \frac{2}{\pi}\int_0^\infty K_1(\tau)\cos(\Omega\tau)d\tau$$
$$= \frac{A^2}{2\pi A_0^2}\{D(2\bar{\sigma}^2 + A^2) + 4 \,\mathrm{Re}\, \hat{C}(-i\Omega)\}, \quad (37)$$

where Re $\hat{C}(-i\Omega)$ is the real part of $\hat{C}(-i\Omega)$, i.e.,

$$\mathrm{Re}\, \hat{C}(-i\Omega) = \int_0^\infty C(t)\cos(\Omega t)dt. \quad (38)$$

Thus

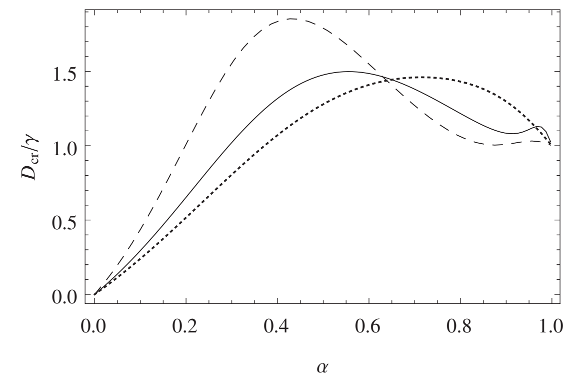

FIG. 1. Critical noise intensity D_{cr} versus the memory exponent α at several values of the parameter γ, [see Eq. (A7)]. All quantities are dimensionless with the time scaling $\omega=1$. Solid line, $\gamma=1.63$; dashed line, $\gamma=4$; dotted line, $\gamma=0.9$.

$$R_{out} = \frac{A_0^2\pi}{D(2\bar{\sigma}^2 + A^2) + 4 \,\mathrm{Re}\, \hat{C}(-i\Omega)}. \quad (39)$$

It follows from Eqs. (30) and (39) that R_{out} decreases monotonically as D increases and thus SR versus the noise intensity D is not possible. It is worth pointing out that R_{out} tends to zero as the noise intensity D tends to the critical value D_{cr}.

III. RESULTS

A. Memory-enhanced energetic stability

In this section we investigate, on the basis of Eq. (A7), the dependence of the appearance of energetic instability on the memory exponent α at various values of the friction coefficient γ. It is well known that in the case of an ordinary oscillator (without memory, $\alpha=1$) with a fluctuating frequency the critical noise intensity $D_{cr}=\gamma\omega^2$. Thus, as soon as the noise intensity exceeds $\gamma\omega^2$, the oscillator becomes energetically unstable [7].

In Fig. 1, several graphs depict the behavior of D_{cr}/γ versus the memory exponent α for different representative values of the parameter γ by the time scaling $\omega=1$. These graphs show a typical resonancelike behavior of $D_{cr}(\alpha)$. As a rule, the maximal value of D_{cr}/γ increases as the value of the friction coefficient γ increases, while the positions of the maxima are monotonically shifted to a lower α as γ rises. Moreover, for some values of γ a multiresonance with two maxima appears [see the solid line in Fig. 1]. Thus, at intermediate values of the memory exponent α the energetic stability of the fractional oscillator is significantly enhanced as compared to an ordinary oscillator (without memory, $\alpha=1$). The effect is very pronounced at high values of the damping γ. A physical explanation for the behavior of $D_{cr}(\alpha)$ in the case of strong memory, $\alpha \to 0$, is based on the cage effect [11]. For small α the friction force induced by the medium is not just slowing down the particle, but also causing the particle to develop a rattling motion. To see this consider Eq. (1) together with Eq. (2) in the limit $\alpha \to 0$,

$$\ddot{X} + [\gamma + \omega^2 + Z(t)]X = \xi(t) + \gamma X(0) + A_0 \sin(\Omega t). \quad (40)$$

Equation (40) describes a stochastically perturbed harmonic motion with a zero-value effective friction coefficient, $\gamma_{ef} = 0$, and with the effective eigenfrequency $\omega_{ef} = \sqrt{\gamma + \omega^2}$. Evidently, the corresponding $D_{cr} = \gamma_{ef}\omega_{ef}^2$ tends to zero. In this sense the medium is binding the particle, preventing dissipation, but forcing oscillations.

To clarify the behavior of $D_{cr}(\alpha)$ at the low memory limit, $\alpha \to 1$, let us take a closer look at the parameter regime $\omega = 1$ and $\gamma < 2$, (cf. Fig. 1). In this case it turns out that, due to the values of the parameters considered, the integral term of the relaxation function $H(t)$ [see Eq. (A1)] at sufficiently low memory, $\alpha \approx 1$, is so small (though it will be dominant at $t \to \infty$) that it can be neglected together with the phaseshift θ. So Eq. (A1) turns to

$$H(t) \approx \frac{2}{\sqrt{u^2 + v^2}} e^{-\beta t} \sin(\omega^* t). \quad (41)$$

To get an idea about the influence of the memory exponent α on the critical noise intensity D_{cr} at $\alpha = 1$ we compare the result Eq. (41) with those from a classical damped oscillator with a fluctuating frequency

$$\ddot{X} + \gamma_{ef}\dot{X} + [\omega_{ef}^2 + Z(t)]X = 0, \quad (42)$$

where

$$\gamma_{ef} = 2\beta, \quad \omega_{ef}^2 = \omega^{*2} + \beta^2. \quad (43)$$

Note, that the relaxation function $H(t)$ for the oscillator Eq. (42) is the same as that presented by Eq. (41). Hence, the critical noise intensity $D_{cr}(\alpha)$ behaves as

$$D_{cr}(\alpha) = \gamma_{ef}\omega_{ef}^2. \quad (44)$$

For the parameter regime $\omega = 1$ and $\gamma < 2$ it follows from Eqs. (A2) and (43) that in the limit $\epsilon \to 0$, where $\epsilon = 1 - \alpha$, the effective parameters $\gamma_{ef}(\epsilon)$ and $\omega_{ef}(\epsilon)$ are increasing functions of ϵ,

$$\left(\frac{d\gamma_{ef}}{d\epsilon}\right)_{|\epsilon=0} = \frac{2\gamma^2}{\sqrt{(4-\gamma^2)}}\left[\pi - \arccos\frac{\gamma}{2}\right] > 0,$$

$$\left(\frac{d\omega_{ef}}{d\epsilon}\right)_{|\epsilon=0} = \frac{1}{\gamma}\left(\frac{d\gamma_{ef}}{d\epsilon}\right)_{|\epsilon=0} > 0. \quad (45)$$

Thus, in the case of very low memory a decrease in the memory exponent causes an increase of the effective eigenfrequency and also an enhancement of dissipation. Consequently, at $\alpha \approx 1$ the critical noise intensity $D_{cr}(\alpha)$ is a decreasing function of α (cf. Fig. 1).

Bearing in mind the behavior of $D_{cr}(\alpha)$ at both of the limiting cases (i.e., at low and strong memory), the occurrence of a maximum of D_{cr} at intermediate values of the memory exponent is not surprising.

B. Memory-induced multiresonance

Next we consider the dependence of two SR characteristics (SNR and variance) on the memory exponent α for various values of the noise intensity D. In order to evaluate the exact behavior of R_{out} and $\bar{\sigma}^2$ vs α, we regard the driving noise $\xi(t)$ in Eq. (1) as an internal noise [see Eqs. (4) and (5)]. In this case the quantities σ_0^2 and Re $\hat{C}(-i\Omega)$ in Eqs. (30) and (39) are given by

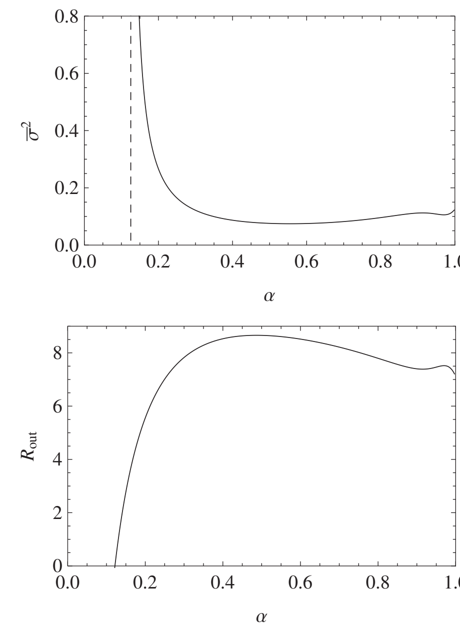

FIG. 2. Dependence of the variance ($\bar{\sigma}^2$) and the output SNR (R_{out}) computed from Eqs. (30), (39), and (46) on the memory exponent α. The parameter values: $\gamma = 1.63$, $A_0 = \Omega = 1$, $k_B T/\omega^2 = 0.01$, and $D = 0.6$. All quantities are dimensionless with the time scaling $\omega = 1$. The dashed line depicts the position of the critical memory exponent $\alpha_1 \approx 0.125$ below which the oscillator is unstable.

$$\sigma_0^2 = \frac{k_B T}{\omega^2}, \quad \text{Re } \hat{C}(-i\Omega) = \frac{k_B T \gamma}{\Omega^{1-\alpha}} \sin\left(\frac{\alpha\pi}{2}\right). \quad (46)$$

In Figs. 2 and 3 we depict, on two panels, the behavior of $\bar{\sigma}^2$ and R_{out} for various values of the noise intensity $D < D_{\max}$, where $D_{\max}$ is the maximal value of $D_{cr}(\alpha)$ by variations of α. Both SR characteristics exhibit a nonmonotonic dependence on the memory exponent α, i.e., a typical resonance phenomenon occurs as α increases. Clearly, the resonancelike behavior of SR characteristics versus α is significantly associated with the nonmonotonous dependence of the critical noise intensity $D_{cr}(\alpha)$ on the memory exponent. In the case considered in Fig. 2 the noise intensity is lower than the critical value at $\alpha = 1$, i.e., $D < \omega^2\gamma$. As a rule (except for some special parameter combinations), in this case the variance $\bar{\sigma}^2$ decreases rapidly from infinity at α_1, $D_{cr}(\alpha_1) = D$, to a minimum, and next increases slowly to a finite value at $\alpha = 1$ as the memory exponent increases. Note that for $\alpha < \alpha_1$ the system is energetically unstable. The dependence of SNR on α corresponds to the behavior of the variance [cf. Eq. (39)] and thus demonstrates two resonancelike maximums in the interval $(\alpha_1, 1)$.

In the case of $\omega^2\gamma < D < D_{\max}$, the phenomenon of memory-induced resonance is very pronounced. Particularly,

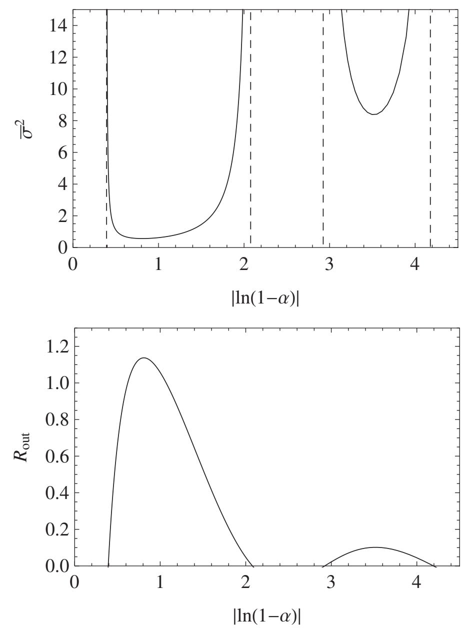

FIG. 3. Multiresonance of SR characteristics [SNR (R_{out}) and variance ($\bar{\sigma}^2$)] versus the memory exponent α at the noise intensity $D=1.8$. The other parameter values are the same as in Fig. 2. The dashed lines depict the position of the critical memory exponents $\alpha_1 \approx 0.324$, $\alpha_2 \approx 0.874$, $\alpha_3 \approx 0.946$, and $\alpha_4 \approx 0.985$, at which $\bar{\sigma}^2$ tends to infinity.

in this case the system is energetically stable only if the values of the memory exponent are in the finite interval $\alpha \in (\alpha_1, \alpha_2)$, $D=D_{cr}(\alpha_{1,2})$; at α_1 and α_2 the variance $\bar{\sigma}^2$ increases unrestrictedly. It is intuitively clear that as long as the variance $\bar{\sigma}^2$, i.e., the initial value of the incoherent part of the output correlation function increases, the noise output spectral density $\Gamma_1(\Omega)$ [see Eq. (37)] increases as well. So the appearance of the resonance peak in R_{out} values is due to a very strong suppression of the SNR by variance amplifications. Hence, the key factor for the appearance of resonance in R_{out} is the occurrence of energetic instability at some values of the memory exponent α.

As mentioned above, for some parameter regimes the dependence of the critical noise intensity on the memory exponent exhibits a double peak structure (see Fig. 1). If $D_{\min} < D < D^*_{\max}$, where $D_{\min}$ and $D^*_{\max}$ are the values of $D_{cr}(\alpha)$ at the local minimum and the local maximum, respectively, this case is characterized by the following scenario (see Fig. 3): For small values of the memory exponent, $\alpha < \alpha_1$, where $D > D_{cr}(\alpha_1)$, the oscillator is energetically unstable. At $\alpha=\alpha_1$, i.e., $D=D_{cr}(\alpha_1)$, the system becomes stable. In the interval $\alpha_1 < \alpha < \alpha_2$ there appears a stable regime, where the SNR exhibits a resonance-like behavior versus α. At $\alpha=\alpha_2$, where $D=D_{cr}(\alpha_2)$, the energetic stability disappears (variance tends to infinity) and the system approaches an unstable regime, thus performing a reentrant transition. With increasing the memory exponent, one observes another region of the values of α, $\alpha_3 < \alpha < \alpha_4$, where the oscillator is energetically stable and the above scenario is repeated. Thus, the occurrence of disjunct stability regions, $\alpha_1 < \alpha < \alpha_2$ and $\alpha_3 < \alpha < \alpha_4$, is manifested in the dependence of the output SNR on α as a phenomenon of memory-induced multiresonance (with two peaks).

Finally, we emphasize that as the critical noise intensity D_{cr} is independent on the driving noise $\xi(t)$, the qualitative picture of the memory-induced resonance for SNR remains unchanged if we replace the internal driving noise $\xi(t)$ by an external noise with a finite σ_0^2 and Re $\hat{C}(-i\Omega)$.

C. Memory-induced energetic stability

Now we will consider the behavior of SR characteristics (R_{out} and $\bar{\sigma}^2$) without the harmonic field, i.e., that of Eq. (1) with a zero eigenfrequency, $\omega=0$. In contrast to the results for SNR in the preceding section, in this case the role of the driving noise $\xi(t)$ is crucial. It is seen from Eqs. (30) and (39) that the time-homogeneous part of the variance of $X(t)$ and the output SNR depend on the quantity σ_0^2, which is determined as the stationary asymptotic value, $t \to \infty$, of the variance of $X(t)$ in the case where the multiplicative fluctuations and the external harmonic force are both absent. The last mentioned particular case [i.e., $\omega = Z(t) = A_0 = 0$ in Eq. (1)] will be referred to in this section as the basic system. It is well known that in the case of internal noise the output process of the basic system is always subdiffusive, i.e., $\sigma_0^2 \sim t^\alpha$, and a stationary regime is impossible [27]. Thus, as the driving noise is internal, the behavior of the system Eq. (1) with $\omega=0$ is subdiffusive and that renders formulas (30) and (39) physically meaningless.

If the driving noise $\xi(t)$ is external, several behaviors of the basic system arise depending on the memory exponent α and the noise correlation function $C(t)$. The exact nature of the leading behavior of σ_0^2 at $t \to \infty$ depends on the exponents of the long-time tails of both the friction kernel and the noise correlation function. The possible behaviors of σ_0^2 include a stationary state, logarithmic growth, subdiffusion, normal diffusion, and superdiffusion [27]. According to [27], we will mention here two simple but general models for the external driving noise $\xi(t)$. (i) Finite noise intensity. In this case it is assumed that the noise intensity defined by

$$D_1 := \int_0^\infty C(t)dt < \infty \tag{47}$$

is finite. (ii) Long-time tail noise. Now the correlation function decays as a power law of the form

$$C(|\tau|) = \langle \xi(t+\tau)\xi(t) \rangle = \frac{D^*}{\Gamma(1-\beta)|\tau|^\beta} \tag{48}$$

with $0 < \beta < 1$ and D^* being a constant. It should be noticed that in the limit $\beta \to 1$ the correlation function presented by Eq. (48) corresponds to a delta-correlated noise, i.e., $C|\tau| = 2D^*\delta(|\tau|)$. In both cases an asymptotic stationary state for the basic model is possible at appropriate values of the memory exponent α. Namely, the variance σ_0^2 is a finite constant at $t \to \infty$ in the case of a finite noise intensity, if $\alpha < 1/2$, and for a long-time tail noise, if $\alpha < \beta/2$ [27]. Thus, in the cases (i) with $\alpha < 1/2$ and (ii) with $\alpha < \beta/2$ the formulas (30) and (39) for the output SNR of the system Eq. (1) without a harmonic trap ($\omega=0$) are applicable. The corresponding relaxation function $H(t)$ and the critical noise intensity D_{cr} are presented in the Appendix with Eqs. (A8) and

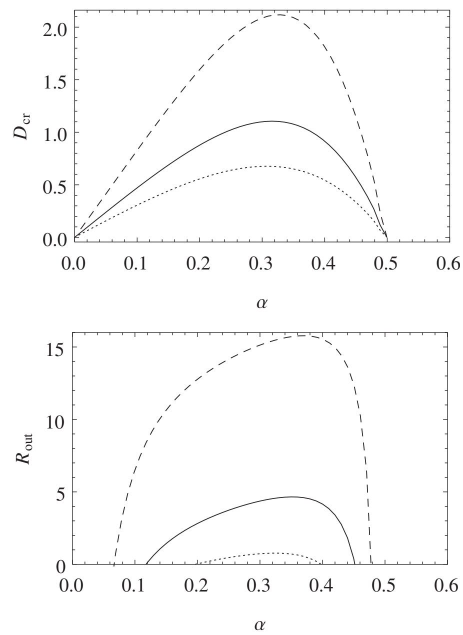

FIG. 4. Critical noise intensity D_{cr} and SNR (R_{out}) as functions of the memory exponent α by absence of the harmonic potential, $\omega=0$, [Eqs. (A11), (39), and (49)]. The parameter values: $A_0=\Omega=1$, $D_1=0.01$, and $D=0.55$. Dashed line, $\gamma=3.0$; solid line, $\gamma=2.1$; dotted line, $\gamma=1.6$. Note that the critical memory exponent $\alpha_{cr}=1/2$.

(A10), respectively. It is remarkable that although the critical intensity of the multiplicative noise D_{cr} [see Eq. (31)] is independent of the driving noise $\xi(t)$, the necessary and sufficient condition for energetic stability is exactly the same as the condition for the existence of a stationary variance σ_0^2 in the case of (i), i.e., $0<\alpha<1/2$. Thus, for $1>\alpha>1/2$ the system is energetically unstable, and for $0<\alpha<1/2$ there is always an external noise $\xi(t)$ for which the system Eq. (1) with $\omega=0$ (unbound system) can exhibit a memory-induced energetic stability. This result of energetic stability for an unbounded system Eq. (1) is highly unexpected, but agrees well with the description of the friction force for small α as an elastic force due to the cage effect (cf. also [11]).

The above described behavior of an unbounded system in the case of a delta-correlated external driving noise with the intensity D_1 is illustrated in Fig. 4. In this case the variance σ_0^2 and Re $\hat{C}(-i\Omega)$ in Eqs. (30) and (39) are given by

$$\sigma_0^2=\frac{D_1}{D_{cr}},\quad \mathrm{Re}\,\hat{C}(-i\Omega)=D_1. \tag{49}$$

The resonant-like behavior of $D_{cr}(\alpha)$ and the SNR versus α, which is seen in Fig. 4, is the manifestation of the cage effect, which is contained in Eq. (1) due to the friction memory kernel.

D. Friction-induced resonance

The main purpose of this section is to demonstrate that a multiresonancelike phenomenon is manifested in the dependence of the SNR for the oscillator Eq. (1) upon the friction

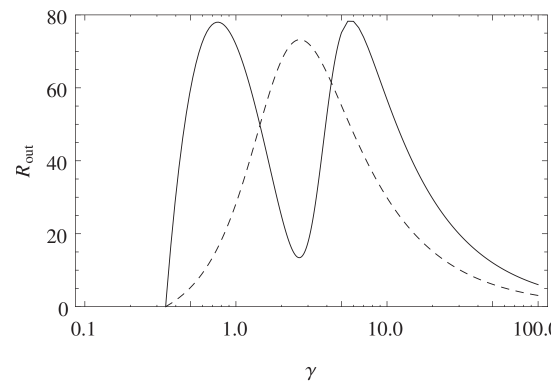

FIG. 5. Dependence of the output SNR (R_{out}) on the friction coefficient γ computed from Eqs. (30), (39), (46), and (A7). The parameter values: $A_0=\omega=1$, $\alpha=0.15$, $k_BT/\omega^2=0.01$, and $D=0.1$. Solid line, $\Omega=2$; dashed line, $\Omega=0.9$. Note that the critical value of the friction coefficient is $\gamma_c=0.342$, at which the output variance tends to infinity.

coefficient γ. The related effect of the friction-induced resonance of the response $A(\gamma)$ to the noisy fractional oscillator with a multiplicative dichotomous noise has previously been reported in [23]. In particular, it was shown that even without a multiplicative noise a friction-induced resonance of $A(\gamma)$ is possible if the conditions

$$\alpha<1,\quad \Omega>\omega \tag{50}$$

are fulfilled. The corresponding position of the resonant maximum is determined by

$$\gamma_m=\frac{1}{\Omega^{\alpha}}(\Omega^2-\omega^2)\cos\left(\frac{\alpha\pi}{2}\right). \tag{51}$$

This result of [23] is somewhat surprising because in the corresponding deterministic models ($\alpha=1$, noise absent), the amplitude of the output oscillations is well known as monotonically decreasing if γ increases.

In Fig. 5 two graphs depict, in the case of the internal noise [see Eqs. (5) and (46)] at $\alpha=0.15$, the behavior of $R_{out}(\gamma)$ for different representative values of the driving frequency Ω. These graphs show a typical resonancelike behavior of $R_{out}(\gamma)$. As a rule, two characteristic regions can be discerned for the driving frequency Ω. (i) In the case of $\Omega<\omega$ there exists one resonance peak of $R_{out}(\gamma)$. (ii) If $\Omega>\omega$, then a multiresonance of $R_{out}(\gamma)$ with two peaks is possible. It is seen from Eqs. (39) and (46) that the formation of the multiresonance versus γ can be explained as an interplay of two different effects: one caused by the energetic instability of the oscillator and another, which is due to the friction-induced resonance of $A(\gamma)$ explained in [23].

Finally, it should be mentioned that the nonmonotonic dependence of R_{out} on the system parameters is not restricted to the dependence on the memory exponent α and on the friction coefficient γ; in some parameter regions also a bona fide resonance of R_{out} versus the frequency Ω of the input signal occurs. As the quantities D_{cr} and σ_0^2 are independent of Ω, it follows from Eq. (39) that the resonancelike behavior of R_{out} on Ω is significantly associated with the resonance of $A(\Omega)$. It is important that for any $\alpha < \alpha_1 \approx 0.441$ and for any values of γ the dependence of $A(\Omega)$ on Ω is always nonmonotonic with a local minimum and with a resonance peak (see [11,23]).

IV. CONCLUSIONS

In this work, we have analyzed the phenomenon of stochastic parametric resonance within the context of a noisy fractional oscillator with a fluctuating eigenfrequency driven by external sinusoidal forcing and by an additive noise [Eq. (1)]. The frequency fluctuations are modeled as a Gaussian white noise. The viscoelastic type friction kernel with memory is assumed as a power-law function of time. The Laplace transformation technique allows us to find exact formulas for the output variance and for the SNR at the long-time limit. A major virtue of the investigated model is that an interplay of parametric fluctuations and the memory of the friction kernel can generate a variety of nonequilibrium cooperation effects.

One of our main results is that the energetic stability of a noisy fractional oscillator can be significantly enhanced at intermediate values of the memory exponent. We have shown that this interesting effect can be interpreted in terms of nonmonotonic dependence of dissipation on the memory of the viscoelastic friction kernel. As another main result we have found a multiresonance of the output SNR versus both the memory exponent and the friction coefficient. We show that these phenomena are significantly associated with the critical characteristics of stochastic parametric resonance. Furthermore, the memory of the friction kernel can induce repeated reentrant transitions between different dynamical regimes of the oscillator. Namely, in some cases an increase of the memory exponent induces transitions from a regime where the system is energetically unstable to a stable regime, but instability appears again through a reentrant transition at higher values of the memory exponent. It is remarkable that in the case of an additive external noise, related phenomena involving memory-induced energetic stability and, associated with that, a resonancelike behavior of SNR versus the memory exponent occurs for the unbound system Eq. (1) (i.e., the harmonic potential is absent). The last mentioned phenomena are a manifestation of the cage effect which is present due to the viscoelastic friction kernel.

It remains to be seen how the reported phenomena can be useful to experimenters in the fields of stochastic resonance, biopolymers, and transient dynamics of molecular and colloidal glasses, where the issues of memory and multiplicative noise can be crucial [13,15,21,22].

Finally, we believe that the results of this paper can also be of interest in some potential technological applications, e.g., for electric oscillator devices with circuit elements of a fractional type (i.e., tree or chain fractances [24]).

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APPENDIX: FORMULAS FOR THE RELAXATION FUNCTION

1. Time dependence of the relaxation function *H*(*t*)

The relaxation function $H(t)$ in Eq. (10) can be obtained by means of the Laplace transformation technique. To evaluate the inverse Laplace transform of

$$\hat{H}(s) = \frac{1}{s^2 + \gamma s^\alpha + \omega^2}$$

[see Eq. (14)] we use the residue theorem method described in [28]. The inverse Laplace transform gives

$$H(t) = \frac{2}{\sqrt{u^2 + v^2}} e^{-\beta t} \sin(\omega^* t + \theta) + \frac{\gamma \sin(\alpha\pi)}{\pi} \int_0^\infty \frac{r^\alpha e^{-rt} dr}{B(r)}, \tag{A1}$$

where $s_{1,2} = -\beta \pm i\omega^*$; $(\beta > 0, \omega^* > 0)$ are the pair of conjugate complex zeros of the equation

$$F(s) \equiv s^2 + \gamma s^\alpha + \omega^2 = 0; \tag{A2}$$

here, $F(s)$ is defined by the principal branch of s^α. The quantities u, v, θ, and $B(r)$ are determined by

$$u = -2\beta + \frac{\alpha\gamma \cos\left[(1-\alpha)\arctan\left(-\frac{\omega^*}{\beta}\right)\right]}{(\beta^2 + \omega^{*2})^{(1-\alpha)/2}}, \tag{A3}$$

$$v = 2\omega^* - \frac{\alpha\gamma \sin\left[(1-\alpha)\arctan\left(-\frac{\omega^*}{\beta}\right)\right]}{(\beta^2 + \omega^{*2})^{(1-\alpha)/2}}, \tag{A4}$$

$$\theta = \arctan\left(\frac{u}{v}\right), \tag{A5}$$

and

$$B(r) := [r^2 + \gamma r^\alpha \cos(\pi\alpha) + \omega^2]^2 + \gamma^2 r^{2\alpha} \sin^2(\alpha\pi). \tag{A6}$$

It must be emphasized that the relaxation function $H(t)$ can be represented via Mittag-Leffler-type special functions [24]. But as the numerical calculations are very complicated, we suggest, apart from possible representations via Mittag-Leffler functions, a numerical treatment of Eq. (A1).

From Eqs. (31) and (A1) one can conclude that the critical noise intensity D_{cr} is given by

$$D_{cr}^{-1} = \frac{2}{\beta(\beta^2+\omega^{*2})[u^2+v^2]^2} \times \{(\beta^2+\omega^{*2})(u^2+v^2) - \beta[\beta(v^2-u^2)-2\omega^* uv]\} + \frac{4\gamma^2\sin^2(\alpha\pi)}{\pi^2}\int_0^\infty \frac{r^\alpha dr}{B(r)}\int_0^r \frac{s^\alpha ds}{(r+s)B(s)} + \frac{8\gamma\sin(\alpha\pi)}{\pi(u^2+v^2)}\int_0^\infty \frac{r^\alpha[v\omega^*+u(r+\beta)]dr}{[\omega^{*2}+(r+\beta)^2]B(r)}. \tag{A7}$$

2. Unbound system

For the unbound system we set $\omega=0$ in Eq. (1). In this case, the relaxation function $H(t)$ can be represented via a Mittag-Leffler-type function as

$$H(t) = tE_{2-\alpha,2}(-\gamma t^{2-\alpha}), \tag{A8}$$

where the generalized Mittag-Leffler function $E_{\eta,\mu}(y)$ is defined by [24]

$$E_{\eta,\mu}(y) = \sum_{n=0}^{\infty}\frac{y^n}{\Gamma(\eta n+\mu)}.$$

As in the case of the harmonically bound system, $\omega\neq 0$, for numerical calculations we prefer an integral representation of the relaxation function, namely

$$H(t) = \frac{2\exp\left[\gamma^{1/(2-\alpha)}t\cos\left(\frac{\pi}{2-\alpha}\right)\right]}{(2-\alpha)\gamma^{1/(2-\alpha)}}\cos\left[t\gamma^{1/(2-\alpha)}\sin\left(\frac{\pi}{2-\alpha}\right)-\frac{\pi}{2-\alpha}\right] + \frac{\gamma\sin(\alpha\pi)}{\pi}\int_0^\infty \frac{e^{-rt}dr}{r^\alpha\tilde{B}(r)}, \tag{A9}$$

where

$$\tilde{B}(r) := r^{2(2-\alpha)} + 2\gamma r^{2-\alpha}\cos(\pi\alpha) + \gamma^2.$$

Thus, from Eqs. (31), (A8), and (A9) we obtain that the critical noise intensity is given by

$$D_{cr}^{-1} = 2\int_0^\infty t^2[E_{2-\alpha,2}(-\gamma t^{2-\alpha})]^2 dt, \tag{A10}$$

or

$$D_{cr}^{-1} = -\frac{2}{\gamma^{3/(2-\alpha)}(2-\alpha)^2\cos\left(\frac{\pi}{2-\alpha}\right)}\left[1+\cos\left(\frac{\pi}{2-\alpha}\right)\cos\left(\frac{3\pi}{2-\alpha}\right)\right] + \frac{2\gamma^2\sin^2(\alpha\pi)}{\pi^2}\int_0^\infty\frac{dr}{r^\alpha\tilde{B}(r)}\int_0^\infty\frac{ds}{(r+s)s^\alpha\tilde{B}(s)} + \frac{8\gamma^{(1-\alpha)/(2-\alpha)}\sin(\alpha\pi)}{\pi(2-\alpha)}\int_0^\infty\frac{\left[r\cos\left(\frac{\pi}{2-\alpha}\right)-\gamma^{1/(2-\alpha)}\cos\left(\frac{2\pi}{2-\alpha}\right)\right]dr}{r^\alpha\tilde{B}(r)\left[r^2-2r\gamma^{1/(2-\alpha)}\cos\left(\frac{\pi}{2-\alpha}\right)+\gamma^{2/(2-\alpha)}\right]}. \tag{A11}$$

Using the asymptotic behavior of the generalized Mittag-Leffler function [24]

$$E_{\eta,\mu}(-y) \approx \frac{1}{y\Gamma(\mu-\eta)}, \quad y\to\infty,$$

it can be seen easily that the integral in Eq. (A10) converges only if $0<\alpha<1/2$, i.e., the critical noise intensity D_{cr} is different from zero only if values of the memory exponent α are in the interval $0<\alpha<1/2$.

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